

# EPIMORPHISMS OF 3-MANIFOLD GROUPS

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**ABSTRACT.** Given a map  $f: M \rightarrow N$  between two 3-manifolds that induces an epimorphism on fundamental groups we show that it is homotopic to a homeomorphism if one of the following holds: either for any finite-index subgroup  $\Gamma$  of  $\pi$  the ranks of  $\Gamma$  and of  $f_*^{-1}(\Gamma)$  agree, or for any finite cover  $\tilde{N}$  of  $N$  the Heegaard genus of  $\tilde{N}$  and the Heegaard genus of the pull-back cover  $\tilde{M}$  agree.

Furthermore we show that fundamental groups of prime 3-manifolds with toroidal boundary are Grothendieck rigid.

## 1. INTRODUCTION

Let  $f: M \rightarrow N$  be a map between two 3-manifolds. (Here and throughout the paper all 3-manifolds are understood to be connected, compact and orientable.) We say  $f$  is *proper* if  $f(\partial M) \subset \partial N$ . We say  $f$  is a  $\pi_1$ -*epimorphism* if the induced map  $f_*: \pi_1(M) \rightarrow \pi_1(N)$  is an epimorphism. Finally  $f$  is called a *degree-one map* if it is proper and if the induced map on homology  $f_*: H_3(M, \partial M; \mathbb{Z}) \rightarrow H_3(N, \partial N; \mathbb{Z})$  is an isomorphism. It is well-known, see e.g. [He76, Lemma 15.12], that a degree-one map is a  $\pi_1$ -epimorphism.

Given a map  $f: M \rightarrow N$  between two 3-manifolds that is either a  $\pi_1$ -epimorphism or a degree-one map it is a long-standing question to find out what extra conditions ensure that  $f$  is in fact homotopic to a homeomorphism. For example, a celebrated theorem of M. Gromov and W. Thurston says that if  $f: M \rightarrow N$  is a degree-one map between two hyperbolic 3-manifolds of the same volume, then  $f$  is homotopic to a homeomorphism, [Th79, Chapter 6], see also [BCG95]. This result has been generalized in many directions, see e.g. [Som95, Der10, Der12], by using the Gromov simplicial volume [Gr82].

For an aspherical 3-manifold  $M$  the volume  $\text{Vol}(M)$  is defined as the sum of the volumes of the hyperbolic pieces of the geometric decomposition of  $M$ . In the following we say that a map  $f: M \rightarrow N$  between manifolds is a  $\mathbb{Z}$ -*homology equivalence*, if  $f$  induces isomorphisms of all homology groups. We recall the following theorem of B. Perron-P. Shalen and P. Derbez.

**Theorem 1.1.** *Let  $f: M \rightarrow N$  be a map between two closed, aspherical 3-manifolds with the same volume. If for any finite covering  $\tilde{N} \rightarrow N$  (not necessarily regular) the induced map  $\tilde{f}: \tilde{M} \rightarrow \tilde{N}$  is a  $\mathbb{Z}$ -homology equivalence, then  $f$  is homotopic to a homeomorphism.*

*Remark.*

- (1) The case of closed graph manifolds was dealt with by B. Perron-P. Shalen [PS99], see also [AF11].
- (2) P. Derbez [Der03] stated the theorem for Haken 3-manifolds that are not graph manifolds. The proof of Derbez applies to all closed 3-manifolds, that are not graph manifolds, for which the Geometrization Conjecture is known. In particular by the work of G. Perelman the theorem applies to all aspherical 3-manifolds that are not graph manifolds.
- (3) The original theorem of P. Derbez is formulated in terms of the simplicial volume, but the simplicial volume for 3-manifolds agrees, up to a fixed constant, with the volume. See e.g. [BP92] for details.
- (4) The conclusion of the theorem does not hold without the assumption on the simplicial volume for in [BW96] an example of two closed aspherical 3-manifolds  $M$  and  $N$  is given, such that for any finite covering  $\tilde{N} \rightarrow N$  the induced map  $\tilde{f}: \tilde{M} \rightarrow \tilde{N}$  is a  $\mathbb{Z}$ -homology equivalence, but such that  $\text{Vol}(M) > \text{Vol}(N)$ .

In the following recall that a subgroup  $\Gamma \subset \pi$  is called *subnormal* if there exists a chain of subgroups  $\pi = \Gamma_0 \supset \Gamma_1 \supset \cdots \supset \Gamma_k = \Gamma$ , such that each  $\Gamma_i$  is normal in  $\Gamma_{i-1}$ . Given a finitely generated group  $\pi$  we denote by  $\text{rk}(\pi)$  its rank, i.e. the smallest cardinality of a generating set of  $\pi$ . If  $f: M \rightarrow N$  is a  $\pi_1$ -epimorphism, then given any finite-index subgroup  $\Gamma$  of  $\pi_1(N)$  the map  $f_*$  induces an epimorphism  $f_*^{-1}(\Gamma) \rightarrow \Gamma$ , in particular the inequality

$$\text{rk}(f_*^{-1}(\Gamma)) \geq \text{rk}(\Gamma)$$

holds. Our first theorem says that equality holds for all finite-index  $\Gamma$ 's only if  $f$  is homotopic to a homeomorphism.

**Theorem 1.2.** *Let  $f: M \rightarrow N$  be a proper map between two aspherical 3-manifolds with empty or toroidal boundary. We assume that  $N$  is not a closed graph manifold. If  $f$  is a  $\pi_1$ -epimorphism and if for every finite-index subnormal subgroup  $\Gamma$  of  $\pi_1(N)$  the equality*

$$\text{rk}((f_*)^{-1}(\Gamma)) = \text{rk}(\Gamma)$$

*holds, then  $f$  is homotopic to a homeomorphism.*

Before we can state our second theorem we need to introduce a few more definitions.

- (1) We say that a covering  $p: \hat{N} \rightarrow N$  of a manifold  $N$  is *subregular* if the covering  $p$  can be written as a composition of coverings  $p_i: N_i \rightarrow N_{i-1}$ ,  $i = 1, \dots, k$  with  $N_k = \hat{N}$  and  $N_0 = N$ , such that each  $p_i$  is regular.
- (2) Given a 3-manifold  $M$  we denote by  $h(M)$  the minimal number of one-handles in a handle-composition of  $M$  with one zero-handle. If  $M$  is closed, then  $h(M)$  equals the *Heegaard genus*, i.e. the minimal genus of a Heegaard surface of  $M$ . If  $M$  has non-empty boundary,  $h(M)$  is smaller or equal to the minimal

genus of a Heegaard splitting of the 3-manifold triad  $(M; \emptyset, \partial M)$  as defined by A. Casson and C. Mc Gordon [CG87], see also M. Scharlemann [Sch02].

Our second theorem gives another variation on Theorem 1.1.

**Theorem 1.3.** *Let  $f: M \rightarrow N$  be a proper map between two aspherical 3-manifolds with empty or toroidal boundary. We assume that  $N$  is not a closed graph manifold. If  $f$  is a  $\pi_1$ -epimorphism and if for every finite subregular cover  $\tilde{N}$  of  $N$  and induced cover  $\tilde{M}$  the inequality  $h(\tilde{M}) = h(\tilde{N})$  holds, then  $f$  is homotopic to a homeomorphism.*

To the best of our knowledge it is not known whether for every  $\pi_1$ -epimorphism  $f: M \rightarrow N$  between two aspherical 3-manifolds the inequality  $h(M) \geq h(N)$  holds. The following result, which is basically a consequence of the proof of Theorem 1.3, shows that the inequality holds virtually.

**Proposition 1.4.** *Let  $f: M \rightarrow N$  be a proper map between two aspherical 3-manifolds with empty or toroidal boundary. We assume that  $N$  is not a closed graph manifold. If  $f$  is a  $\pi_1$ -epimorphism, then there exists a finite subregular cover  $\tilde{N}$  of  $N$  such that the induced cover  $\tilde{M}$  satisfies the inequality  $h(\tilde{M}) \geq h(\tilde{N})$ .*

*Remark.* In Theorems 1.2, 1.3 and in Proposition 1.4 we excluded the case that  $N$  is a closed graph manifold. What we really need for the statements to hold is that  $N$  is a 3-manifold that is virtually fibered. By the Virtual Fiber Theorem of Agol [Ag08, Ag13] Przytycki–Wise [PW14, PW12] and Wise [Wi09, Wi12a, Wi12b] any aspherical 3-manifold that is *not* a closed graph manifold is virtually fibered. The three results also apply if  $N$  is a closed graph manifold that is virtually fibered, e.g. if  $N$  is a Sol-manifold or if  $N$  carries a Riemannian metric of nonpositive sectional curvature [Liu013]. We do not know though whether our results hold for aspherical graph manifolds that are not virtually fibered. We refer to [LW93, Ne96] for more examples of graph manifolds that are not virtually fibered.

Now we turn to the study of profinite completions of the fundamental groups of 3-manifolds. Following [LR11] we say that a group  $\pi$  is *Grothendieck rigid* if for every finitely generated proper subgroup  $\Gamma \subset \pi$  the inclusion induced map  $\hat{\Gamma} \rightarrow \hat{\pi}$  of profinite completions is not an isomorphism. Cavendish [Ca12], see also Reid [Rei15, Theorem 8.3], showed that the fundamental groups of prime closed 3-manifolds are Grothendieck rigid. We extend this result to prime 3-manifolds with toroidal boundary. More precisely, we have the following theorem.

**Theorem 1.5.** *The fundamental group of any prime 3-manifold with empty or toroidal boundary is Grothendieck rigid.*

The following consequence is again more in spirit with the earlier results of the paper.

**Corollary 1.6.** *Let  $f: M \rightarrow N$  be a proper map between two aspherical 3-manifolds with empty or toroidal boundary. If  $f$  is a  $\pi_1$ -epimorphism and if there exists an*

isomorphism  $\widehat{\pi_1(M)} \cong \widehat{\pi_1(N)}$  (not necessarily induced by  $f$ ), then  $f$  is homotopic to a homeomorphism.

This paper is organized as follows. In Section 2 we provide the proof of Theorem 1.2. The proof of Theorem 1.3 has many formal similarities with the proof of Theorem 1.3. In Section 3 we will point out how to modify the proof of Theorem 1.2 to obtain a proof of Theorem 1.3. Finally in Section 4 we turn to the profinite completions of 3-manifold groups. We recall the definition and basic properties of profinite completions in Section 4.1. In Section 4.2 we prove Theorem 1.5 for 3-manifolds with non-empty boundary. The case of closed 3-manifolds has been dealt with by Cavendish [Ca12]. We conclude the paper with a proof of Corollary 1.6 in Section 4.3.

**Conventions.** Throughout this paper all 3-manifolds are understood to be connected, compact and orientable.

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## 2. THE RANK OF FUNDAMENTAL GROUPS AND $\pi_1$ -EPIMORPHISMS

**2.1. Proof of Theorem 1.2 in the fibered case.** Let  $M$  be a 3-manifold. Throughout this paper we identify  $H^1(M; \mathbb{Z})$  with  $\text{Hom}(\pi_1(M), \mathbb{Z})$ . We say a primitive class  $\phi \in H^1(M; \mathbb{Z}) = \text{Hom}(\pi_1(M), \mathbb{Z})$  is *fibered* if there exists a fibration  $p: M \rightarrow S^1$  such that the induced map  $p_*: \pi_1(M) \rightarrow \pi_1(S^1) = \mathbb{Z}$  coincides with  $\phi$ . It follows from Stallings’ theorem [St62] (together with the resolution of the Poincaré conjecture) that  $\phi$  is a fibered class if and only if  $\ker(\phi: \pi_1(M) \rightarrow \mathbb{Z})$  is finitely generated.

**Proposition 2.1.** *Suppose that  $f: M \rightarrow N$  is a proper  $\pi_1$ -epimorphism between two 3-manifolds. We assume that there exists a primitive class  $\phi \in H^1(N; \mathbb{Z})$  such that  $\phi \in H^1(N; \mathbb{Z})$  and  $f^*\phi \in H^1(M; \mathbb{Z})$  are fibered. If  $\text{rk}(\ker(f^*\phi)) = \text{rk}(\ker(\phi))$ , then  $f_*: \pi_1(M) \rightarrow \pi_1(N)$  is an isomorphism.*

*Proof.* We denote by  $F$  the fiber of  $\phi$  and we denote by  $E$  the fiber of  $f^*\phi$ . These are compact orientable surfaces.

We consider the following commutative diagram of short exact sequences

$$\begin{array}{ccccccc} 1 & \longrightarrow & \pi_1(E) & \longrightarrow & \pi_1(M) & \xrightarrow{\phi \circ f_*} & \mathbb{Z} \longrightarrow 0 \\ & & \downarrow f_* & & \downarrow f_* & & \downarrow = \\ 1 & \longrightarrow & \pi_1(F) & \longrightarrow & \pi_1(N) & \xrightarrow{\phi} & \mathbb{Z} \longrightarrow 0. \end{array}$$

Since  $f_*: \pi_1(M) \rightarrow \pi_1(N)$  is an epimorphism it follows that the map on the left is also an epimorphism. By our hypothesis we have

$$\text{rk}(\pi_1(E)) = \text{rk}(\ker(f^*\phi)) = \text{rk}(\ker(\phi)) = \text{rk}(\pi_1(F)).$$

We make the following claim.

*Claim.* The induced map  $\pi_1(E) \rightarrow \pi_1(F)$  is an isomorphism.

First consider the case that  $M$  is closed. In that case  $E$  is also closed. Together with the observation that  $f_*: \pi_1(E) \rightarrow \pi_1(F)$  is an epimorphism and from the observation that  $\text{rk}(\pi_1(E)) = \text{rk}(\pi_1(F))$  we deduce that  $F$  is also closed and that the map  $f_*: \pi_1(E) \rightarrow \pi_1(F)$  is an isomorphism.

Now we consider the case that  $M$  has boundary. Then also  $E$  has boundary. Since  $f$  is a proper map the manifold  $N$ , and thus also  $F$  have boundary. Thus  $f_*: \pi_1(E) \rightarrow \pi_1(F)$  is an epimorphism between free groups of the same rank, thus it is already an isomorphism. This concludes the proof of the claim.

An elementary argument using the above commutative diagram shows that the induced map  $f_*: \pi_1(M) \rightarrow \pi_1(N)$  is an isomorphism.  $\square$

## 2.2. Ranks and fibers.

**Proposition 2.2.** *Let  $M$  be a 3-manifold. We write  $\pi = \pi_1(M)$ . Furthermore let  $\phi \in H^1(M; \mathbb{Z}) = \text{Hom}(\pi, \mathbb{Z})$  be a fibered class. We write*

$$\pi_n = \ker(\pi_1(M) \xrightarrow{\phi} \mathbb{Z} \rightarrow \mathbb{Z}/n).$$

*Let  $p$  be a prime. For any  $n$  we have*

$$\dim_{\mathbb{F}_p}(H_1(\pi_n; \mathbb{F}_p)) \leq \text{rk}(\pi_n) \leq 1 + \text{rk}(\ker(\phi)).$$

*Furthermore, there exists an  $n$  such that*

$$\dim_{\mathbb{F}_p}(H_1(\pi_n; \mathbb{F}_p)) = \text{rk}(\pi_n) = 1 + \text{rk}(\ker(\phi)).$$

*Proof.* We denote by  $F$  the fiber of the fibration corresponding to  $\phi$  and we write  $\Gamma = \pi_1(F)$ . We can identify  $\pi$  with a semidirect product  $\langle t \rangle \rtimes_{\mu} \Gamma$  in such a way that  $\phi(t) = 1$  and  $\phi|_{\Gamma}$  is trivial. Here  $\mu: \Gamma \rightarrow \Gamma$  denotes the corresponding automorphism of  $\Gamma$ , which is just given by the monodromy action on  $\Gamma = \pi_1(F)$ .

Given any  $n$  we have

$$\pi_n = \langle t^n \rangle \rtimes \Gamma.$$

It follows that

$$\dim_{\mathbb{F}_p}(H_1(\pi_n; \mathbb{F}_p)) \leq \text{rk}(\pi_n) \leq 1 + \text{rk}(\Gamma) = 1 + \text{rk}(\ker(\phi)).$$

Now let  $p$  be a prime. The homology group  $H_1(\Gamma; \mathbb{F}_p)$  is finite. Thus there exists an  $n$  such that  $\mu_*^n$  acts like the identity on  $H_1(\Gamma; \mathbb{F}_p)$ . It follows that

$$\begin{aligned} \text{rk}(\pi_n) &= \text{rk}(\langle t^n \rangle \rtimes_{\mu} \Gamma) = \text{rk}(\langle t \rangle \rtimes_{\mu^n} \Gamma) \geq H_1(\langle t \rangle \rtimes_{\mu^n} \Gamma; \mathbb{F}_p) \\ &= \text{rk}(\mathbb{Z} \oplus H_1(\Gamma; \mathbb{F}_p)) \\ &= 1 + \text{rk}(\Gamma) = 1 + \text{rk}(\ker(\phi)). \end{aligned}$$

Here we used that for the fundamental group of the orientable compact surface  $F$  we have  $\text{rk}(\pi_1(F)) = \dim_{\mathbb{F}_p}(H_1(F; \mathbb{F}_p))$ .  $\square$

**2.3. The rank gradient and fiberedness.** Let  $\pi$  be a group and  $\phi: \pi \rightarrow \mathbb{Z}$  be a homomorphism. We write

$$\pi_n = \ker(\pi \xrightarrow{\phi} \mathbb{Z} \rightarrow \mathbb{Z}/n).$$

Following [La05] and [DFV14] we refer to

$$\text{rg}(\pi, \phi) := \liminf_{n \rightarrow \infty} \frac{1}{n} \text{rk}(\pi_n)$$

as the *rank gradient* of  $(\pi, \phi)$ .

**Theorem 2.3.** *Let  $M$  be a 3-manifold and let  $\phi: \pi_1(M) \rightarrow \mathbb{Z}$  be a non-trivial homomorphism. Then the following two statements are equivalent:*

- (1)  $\phi \in \text{Hom}(\pi_1(M), \mathbb{Z}) = H^1(M; \mathbb{Z})$  is fibered,
- (2)  $\text{rg}(\pi_1(M), \phi) = 0$ .

Here the implication (1)  $\Rightarrow$  (2) is an immediate consequence of Proposition 2.2. The implication (2)  $\Rightarrow$  (1) is the main result of [DFV14] (see also [De13]). The proof in [DFV14] builds on the fact that twisted Alexander polynomials detect fibered manifolds [FV12], that proof in turn relies on the recent results of Wise [Wi09, Wi12a, Wi12b].

**2.4. Proof of Theorem 1.2.** Before we provide a proof of Theorem 1.2 we recall the Virtual Fiber Theorem of Agol [Ag08, Ag13], Przytycki–Wise [PW14, PW12] and Wise [Wi09, Wi12a, Wi12b]. (We refer to [AFW15] for the precise references for the Virtual Fiber Theorem.)

**Theorem 2.4.** *Any prime manifold that is not a closed graph manifold admits a finite cover that is fibered.*

Now we are finally in a position to prove Theorem 1.2.

*Proof of Theorem 1.2.* Suppose that  $f: M \rightarrow N$  is a proper  $\pi_1$ -epimorphism between two aspherical 3-manifolds with empty or toroidal boundary. We assume  $N$  is not a closed graph manifold. We suppose that the following condition holds:

- (\*) Given any finite-index subnormal subgroup  $\Gamma$  of  $\pi_1(M)$  the equality

$$\text{rk}(f_*^{-1}(\Gamma)) = \text{rk}(\Gamma)$$

holds.

We will show that  $f$  is homotopic to a homeomorphism. Since  $N$  is a prime 3-manifold with empty or toroidal boundary that is not a closed graph manifold it follows from Theorem 2.4 that  $N$  admits a finite-index regular covering  $p: \tilde{N} \rightarrow N$  such that  $\tilde{N}$  admits a primitive fibered class  $\phi \in H^1(\tilde{N}; \mathbb{Z}) = \text{Hom}(\pi_1(\tilde{N}), \mathbb{Z})$ .

We denote by  $\tilde{M}$  the finite-cover of  $M$  corresponding to  $\pi_1(\tilde{M}) = f_*^{-1}(\pi_1(\tilde{N}))$ . Given  $n \in \mathbb{N}$  we denote by  $\tilde{M}_n$  the cover of  $\tilde{M}$  corresponding to the epimorphism

$$\pi_1(\tilde{M}) \xrightarrow{\phi \circ f_*} \mathbb{Z} \rightarrow \mathbb{Z}/n.$$

Similarly we denote by  $\widetilde{N}_n$  the cover of  $\widetilde{N}$  corresponding to the epimorphism

$$\pi_1(\widetilde{N}) \xrightarrow{\phi} \mathbb{Z} \rightarrow \mathbb{Z}/n.$$

Evidently  $\pi_1(\widetilde{N}_n)$  is a subnormal subgroup of  $\pi_1(N)$  and we have  $\pi_1(\widetilde{M}_n) = f_*^{-1}(\pi_1(\widetilde{N}_n))$ . By our assumption (\*) we have

$$\text{rk}(\pi_1(\widetilde{M}_n)) = \text{rk}(\pi_1(\widetilde{N}_n)) \text{ for all } n.$$

Since  $\phi$  is fibered it follows from Theorem 2.3 that  $\text{rg}(\pi_1(\widetilde{N}), \phi) = 0$ . By the above observation we have  $\text{rg}(\pi_1(\widetilde{M}), f^*\phi) = 0$ . It is a consequence of Theorem 2.3 that  $f^*\phi$  is a fibered class of  $\widetilde{M}$ .

Similarly, it is a consequence of the above observations and of Proposition 2.2 that  $\text{rk}(\ker(f^*\phi)) = \max \{\text{rk}(\pi_1(\widetilde{M})_n) \mid n \in \mathbb{N}\} = \max \{\text{rk}(\pi_1(\widetilde{N}_n)) \mid n \in \mathbb{N}\} = \text{rk}(\ker(\phi))$ .

It is now a consequence of Proposition 2.1 that  $f_*: \pi_1(\widetilde{M}) \rightarrow \pi_1(\widetilde{N})$  is an isomorphism.

Next we show that  $f_*: \pi_1(M) \rightarrow \pi_1(N)$  is already an isomorphism. We consider the following diagram

$$\begin{array}{ccccccc} 1 & \longrightarrow & \pi_1(\widetilde{M}) & \longrightarrow & \pi_1(M) & \longrightarrow & \pi_1(M)/\pi_1(\widetilde{M}) \longrightarrow 1 \\ & & \cong \downarrow f_* & & \downarrow f_* & & \cong \downarrow f_* \\ 1 & \longrightarrow & \pi_1(\widetilde{N}) & \longrightarrow & \pi_1(N) & \longrightarrow & \pi_1(N)/\pi_1(\widetilde{N}) \longrightarrow 1. \end{array}$$

This diagram consists of maps between pointed sets (i.e. sets with distinguished elements). The top and bottom sequence are exact in the category of pointed sets. The left and the rightmost maps are isomorphisms of pointed sets. The diagram evidently commutes. Thus it follows from the five-lemma that the middle map is also an isomorphism.

Summarizing we showed that  $f_*: \pi_1(M) \rightarrow \pi_1(N)$  is an isomorphism. Since  $f: (M, \partial M) \rightarrow (N, \partial N)$  is a proper map, it follows from work of Waldhausen [Wa68, Corollary 6.5][He76, Theorem 13.16] and from Mostow-Prasad rigidity [Mo68, Pr73] that  $f$  is homotopic to a homeomorphism.  $\square$

### 3. THE HEEGAARD GENUS AND $\pi_1$ -EPIMORPHISMS

**3.1. Proof of Theorem 1.3.** In this section we will provide the proof of Theorem 1.3. We start out with the following well-known lemma.

**Lemma 3.1.** *Let  $M$  be a manifold. Then the following hold:*

- (1)  $\text{rk}(\pi_1(M)) \leq h(M)$ .
- (2) *If  $M$  is a fibered 3-manifold with connected fiber  $F$ , then*

$$h(M) \leq 2g(F) + 1.$$

*Proof.* By definition we can endow  $M$  with a handle decomposition with one zero-cell and  $h(M)$  one-cells. It is thus clear that the free group on  $h(M)$  generators surjects onto  $\pi_1(M)$ . Therefore  $\text{rk}(\pi_1(M)) \leq h(M)$ . This concludes the proof of (1).

Now we turn to the proof of (2). If  $M$  is fibered with fiber  $F$ , then we can write  $M = F \times [0, 4]/(x, 0) \sim (\varphi(x), 4)$ . First we consider the case that  $M$  is closed. Pick two closed disks  $P \subset \Sigma$  and  $Q \subset \Sigma$  with  $P \cap Q = \emptyset$  and  $\varphi(Q) \cap P = \emptyset$ . It is straightforward to see that

$$(\overline{F \setminus (P \cup Q)}) \times \{0\} \cup \partial P \times [0, 2] \cup (\overline{F \setminus (P \cup \varphi(Q))}) \times \{2\} \cup \partial \varphi(Q) \times [2, 4]$$

splits  $M$  into two handlebodies of genus  $2g(F) + 1$ . It follows that we can endow  $M$  with a handle-decomposition with one zero-handle and  $2g + 1$  one-handles. Now we consider the case that  $M$  has non-empty boundary. We pick a closed disk  $P \subset F$ . Then the image of  $F \times [0, 1] \cup P \times [1, 3] \cup F \times [3, 4]$  in  $M$  can be written as a handlebody with one 0-cell and  $2g + 1$  1-cells. Furthermore  $M$  can be built out of this handlebody by attaching 2-cells and 3-cells. It follows that  $h(M) \leq 2g(F) + 1$ .  $\square$

The following proposition plays the rôle of Proposition 2.2.

**Proposition 3.2.** *Let  $M$  be a 3-manifold. Furthermore let  $\phi \in H^1(M; \mathbb{Z}) = \text{Hom}(\pi, \mathbb{Z})$  be a primitive fibered class with corresponding fiber  $F$ . Given  $n \in \mathbb{N}$  we denote by  $M_n$  the cover of  $M$  corresponding to  $\ker(\pi_1(M) \xrightarrow{\phi} \mathbb{Z} \rightarrow \mathbb{Z}/n)$ . For any  $n$  we have*

$$h(M_n) \leq 2g(F) + 1.$$

Furthermore, there exists an  $n$  such that

$$h(M_n) = 2g(F) + 1.$$

*Remark.* If  $M$  is closed and hyperbolic, then Souto [Sou08, Theorem 1.1] showed that there exists an  $m$  such that for every  $n \geq m$  we have  $h(M_n) = 2g(F) + 1$ .

*Proof.* We pick a fibration  $p: M \rightarrow S^1$  with fiber  $F$  and monodromy  $\varphi: F \rightarrow F$  that corresponds to the given primitive fibered class  $\phi$ . Since  $\phi$  is primitive the fiber  $F$  is connected. For any  $n$  the fibration  $p$  gives rise to a fibration  $p_n: M_n \rightarrow S^1$  with fiber  $F$  and monodromy  $\varphi^n$ . The first statement is now a consequence of Lemma 3.1.

By Proposition 2.2 there exists an  $n$  such that

$$\text{rk}(\pi_1(M_n)) = 1 + \text{rk}(\pi_1(F)).$$

It follows from Lemma 3.1 that

$$1 + \text{rk}(\pi_1(F)) = \text{rk}(\pi_1(M_n)) \leq h(M_n) \leq 1 + 2g(F) = 1 + \text{rk}(\pi_1(F)).$$

Thus we obtain that  $h(M_n) = 2g(F) + 1$ .  $\square$

Let  $M$  be a 3-manifold and let  $\phi: \pi_1(M) \rightarrow \mathbb{Z}$  be an epimorphism. We write  $M_n$  for the finite cyclic cover corresponding to

$$\ker(\pi_1(M) \xrightarrow{\phi} \mathbb{Z} \rightarrow \mathbb{Z}/n).$$



Following [La06] we refer to

$$\mathrm{hg}(M, \phi) := \liminf_{n \rightarrow \infty} \frac{1}{n} (2h(M_n) - 2)$$

as the *Heegaard gradient* of  $(M, \phi)$ . The following proposition plays the rôle of Theorem 2.3.

**Proposition 3.3.** *Let  $M$  be a 3-manifold and let  $\phi: \pi_1(M) \rightarrow \mathbb{Z}$  be an epimorphism. Then the following two statements are equivalent:*

- (1)  $\phi \in \mathrm{Hom}(\pi_1(M), \mathbb{Z}) = H^1(M; \mathbb{Z})$  is fibered,
- (2)  $\mathrm{hg}(M, \phi) = 0$ .

Here the implication (1)  $\Rightarrow$  (2) is an immediate consequence of Proposition 3.2. For the implication (2)  $\Rightarrow$  (1) note that if  $\mathrm{hg}(M, \phi) = 0$ , then it follows from Lemma 3.1 that  $\mathrm{rg}(M, \phi) = 0$ , which in turn implies by Theorem 2.3 that  $\phi$  is fibered.

*Remark.* The implication (2)  $\Rightarrow$  (1) was first proved by M. Lackenby [La06, Theorem 1.11] for closed hyperbolic 3-manifolds (see also [Ren10, Ren14] for extensions). The proof of M. Lackenby is significantly harder than the proof provided in [DFV14] since the latter proof relies on the recent results of Wise [Wi09, Wi12a, Wi12b].

The proof of Theorem 1.3 is now almost entirely identical to the proof of Theorem 1.2, we only need to replace the study of ranks of subnormal finite-index subgroups of  $\pi_1(M)$  and  $\pi_1(N)$  by the Heegaard genera of the corresponding subregular finite-index covers of  $M$  and  $N$ . In particular we need to replace Proposition 2.2 by Proposition 3.2 and we need to replace Theorem 2.3 by Proposition 3.3. We leave the straightforward and dull details to the reader.

**3.2. Proof of Proposition 1.4.** Let  $f: M \rightarrow N$  be a degree-one map between two aspherical 3-manifolds with empty or toroidal boundary. We assume that  $N$  is not a closed graph manifold. As in the proof of Theorem 1.2 there exists a finite subregular cover  $\tilde{N}$  of  $N$  which is fibered with fiber  $F$  and which satisfies

$$\mathrm{rk}(\pi_1(\tilde{M})) = 1 + 2 \mathrm{rk}(\pi_1(F)).$$

We denote by  $\tilde{M}$  the induced cover of  $M$ . The map  $f$  gives rise to a degree-one map  $\tilde{f}: \tilde{M} \rightarrow \tilde{N}$ , which induces in particular an epimorphism of fundamental groups. It follows from Lemma 3.1 that

$$h(\tilde{M}) \geq \mathrm{rk}(\pi_1(\tilde{M})) \geq \mathrm{rk}(\pi_1(\tilde{N})) = 1 + 2 \mathrm{rk}(\pi_1(F)) = 1 + 2g(F) \geq h(\tilde{N}).$$

This concludes the proof of Proposition 1.4.

#### 4. THE PROFINITE COMPLETION OF THE FUNDAMENTAL GROUP OF A 3-MANIFOLD

**4.1. The profinite completion of a group.** In this section we recall several basic properties of profinite completions. Throughout this section we refer to [RZ10] for details. Given a group  $\pi$  we consider the inverse system  $\{\pi/\Gamma\}_\Gamma$  where  $\Gamma$  runs over all finite index normal subgroups of  $\pi$ . The profinite completion  $\widehat{\pi}$  of  $\pi$  is then defined as the inverse limit of this system, i.e.

$$\widehat{\pi} = \varprojlim \pi/\Gamma.$$

Note that the natural map  $\pi \rightarrow \widehat{\pi}$  is injective if and only if  $\pi$  is residually finite. It follows from [He87] and the proof of the Geometrization Conjecture that fundamental groups of 3-manifolds are residually finite.

The following lemma is consequence of a deep result of N. Nikolov and D. Segal [NS07]. We refer to [RZ10, Proposition 3.2.2] for details.

**Lemma 4.1.** *Let  $\pi$  be a finitely generated residually finite group. Then for every finite group  $G$  the map  $\pi \rightarrow \widehat{\pi}$  induces a bijection  $\text{Hom}(\widehat{\pi}, G) \rightarrow \text{Hom}(\pi, G)$ .*

**Lemma 4.2.** *If  $f: A \rightarrow B$  is a homomorphism between two finitely generated residually finite groups that induces an isomorphism of profinite completions, then  $f_*: H_1(A; \mathbb{Z}) \rightarrow H_1(B; \mathbb{Z})$  is an isomorphism.*

*Proof.* It follows from Lemma 4.1 that the map  $f$  induces for every finite abelian group  $G$  a bijection  $\text{Hom}(A, G) \rightarrow \text{Hom}(B, G)$ . It now follows easily from the classification of finitely generated abelian groups that  $f_*: H_1(A; \mathbb{Z}) \rightarrow H_1(B; \mathbb{Z})$  is an isomorphism.  $\square$

**Lemma 4.3.** *Let  $f: \widehat{A} \rightarrow \widehat{B}$  be an isomorphism between the profinite completions of two finitely generated residually finite groups  $A$  and  $B$ . If  $\beta: B \rightarrow G$  is a homomorphism to a finite group, then  $f$  restricts to an isomorphism*

$$\widehat{\ker(\beta \circ f)} \rightarrow \widehat{\ker(\beta)}.$$

*Proof.* First note that if  $\gamma: C \rightarrow G$  is an epimorphism onto a finite group, then  $\widehat{\ker(\gamma)} = \ker(\gamma: \widehat{C} \rightarrow G)$ . Now let  $\beta: B \rightarrow G$  be a homomorphism to a finite group. We write  $\alpha = \beta \circ f$ . We then obtain the following commutative diagram

$$\begin{array}{ccccccc} 1 & \longrightarrow & \ker(\alpha: \widehat{A} \rightarrow G) & \longrightarrow & \widehat{A} & \longrightarrow & G \longrightarrow 1 \\ & & \downarrow f & & \downarrow f & & \downarrow \\ 1 & \longrightarrow & \ker(\alpha: \widehat{B} \rightarrow G) & \longrightarrow & \widehat{B} & \longrightarrow & G \longrightarrow 1. \end{array}$$

Since the maps in the middle and on the right are isomorphisms it now follows that the map on the left is an isomorphism. The lemma now follows from the fact, mentioned at the beginning of the proof, that the groups on the left agree with the profinite completions of the kernels of  $\alpha: A \rightarrow G$  and  $\beta: B \rightarrow G$ .  $\square$

Given a group  $\pi$  we denote by  $Q(\pi)$  the set of finite quotients of  $\pi$ . By [RZ10, Corollary 3.2.8] the following lemma holds.

**Lemma 4.4.** *Two finitely generated residually finite groups  $A$  and  $B$  have isomorphic profinite completions if and only if  $Q(A) = Q(B)$ .*

#### 4.2. Grothendieck rigidity of 3-manifolds with toroidal boundary.

*Definition.* A residually finite group  $\pi$  is called *Grothendieck rigid* if for a finitely generated subgroup  $\Gamma \subset \pi$  the induced map on profinite completions  $\widehat{\Gamma} \rightarrow \widehat{\pi}$  is an isomorphism if and only if  $\Gamma = \pi$ .

Long–Reid [LR11] showed that fundamental groups of hyperbolic 3-manifolds are Grothendieck rigid. Cavendish [Ca12, Proposition 3.7.1] and Reid [Rei15, Theorem 8.3] showed that fundamental groups of closed prime 3-manifolds are Grothendieck rigid. In this section we prove Theorem 1.5 which settles the case of prime 3-manifolds with toroidal boundary.

Before we start proving Theorem 1.5 we recall that, following Serre [Se97, D.2.6. Exercise D] a group  $\pi$  is called *good* if for every finite abelian group  $A$  and any representation  $\alpha: \pi \rightarrow \text{Aut}(A)$  and any  $i$  the natural map

$$H^i(\widehat{\pi}; A) \rightarrow H^i(\pi; A)$$

is an isomorphism.

The following theorem is due to the aforementioned work of Wise [Wi09, Wi12a, Wi12b], to work of Wilton–Zaleskii [WZ10] and Cavendish [Ca12, Section 3.5 and Lemma 3.7.1]. We also refer to [AFW15, (H.26)] for details.

**Theorem 4.5.** *The fundamental group of any 3-manifold is good.*

**Proof of Theorem 1.5.** Let  $M$  be a prime 3-manifold with toroidal boundary and let  $\Gamma \subset \pi := \pi_1(M)$  be a finitely generated subgroup with  $\Gamma \neq \pi$ . We need to show that the inclusion induced map  $\widehat{\Gamma} \rightarrow \widehat{\pi}$  is not an isomorphism.

First suppose that  $\Gamma$  is a finite-index subgroup. Then it follows from standard arguments, see e.g. [LR11, Lemma 2.5] that  $\widehat{\Gamma} \rightarrow \widehat{\pi}$  is not an isomorphism.

Now suppose that  $\Gamma$  is an infinite-index subgroup. By Scott’s core theorem [Sco73] there exists a 3-dimensional submanifold  $N \subset M$  with  $\pi_1(N) = \Gamma$ . (Here the 3-manifold  $N$  can a priori have any type of boundary.) Since  $M$  is prime and since it has non-trivial boundary we know that  $M$  is irreducible. This allows us to cap off all boundary components of  $N$  that are 2-spheres. Therefore we can assume that all boundary components of  $N$  have genus at least one.

Suppose  $N$  has a boundary component that is not a torus. It follows that  $\chi(N) = \chi(\pi_1(N)) \neq 0$ , whereas  $\chi(M) = \chi(\pi_1(M)) = 0$ . By Theorem 4.5 the groups  $\pi_1(M)$  and  $\pi_1(N)$  are good. If we apply the goodness property to  $A = \mathbb{F}_p$  with the trivial action we obtain that  $\chi(\widehat{\pi_1(M)}) = \chi(\pi_1(M)) \neq 0$  and  $\chi(\widehat{\pi_1(N)}) = \chi(\pi_1(N)) = 0$ . Thus the inclusion induced map  $\widehat{\Gamma} \rightarrow \widehat{\pi}$  can not be an isomorphism. (An alternative

argument is given as follows: if the inclusion  $\Gamma \rightarrow \pi$  induces an isomorphism  $\widehat{\Gamma} \xrightarrow{\cong} \widehat{\pi}$ , then it follows from the Lück approximation theorem [Lü94] that  $M$  and  $N$  have the same  $L^2$ -Betti numbers, but this is not the case by the calculation given in [Lü02].)

Finally suppose that all boundary components of  $N$  are tori. By Lemma 4.3 and Lemma 4.2 it suffices to prove the following claim.

*Claim.* There exists a finite-index subgroup  $\pi' \subset \pi$  such that for  $\Gamma' := \Gamma \cap \pi'$  the induced map  $H_1(\Gamma'; \mathbb{Z}) \rightarrow H_1(\pi'; \mathbb{Z})$  is not an isomorphism.

By the Characteristic Pair Theorem [JS79, p. 138] we can assume that each boundary torus of  $N$  is of one of the following types:

- (1) a boundary component of  $M$ ,
- (2) a JSJ torus of  $M$ ,
- (3) a torus embedded in a Seifert fibered piece of the JSJ-decomposition of  $M$  that is not isotopic to a boundary component of  $M$  and not isotopic to a JSJ torus of  $M$ .

By [Ja80, Theorem VI.34] we can assume that the tori of the third type are vertical, i.e. given by a union of fibers in the Seifert fibered structure of the JSJ component. Now denote by  $T = \{T_1, \dots, T_k\}$  the collection of tori that is given by the JSJ tori of  $M$  together with the boundary tori of  $N$  of the third type. We denote by  $W_1, \dots, W_l$  the components of  $M$  cut along the tori in  $T$ . Note that each  $W_i$  is either hyperbolic or Seifert fibered. The submanifold  $N$  is given by gluing together some of the components  $W_i$ .

Since  $\Gamma$  is a proper subgroup of  $\pi$ , since  $N$  is a connected submanifold and since none of the  $W_i$ 's is a product  $T \times I$  we know that one of the following occurs:

- (1)  $N$  does not contain a  $T_i$  that is non-separating
- (2)  $N$  does not contain one of the  $W_i$ 's.

In the first case there exists a curve  $c$  in  $M$  which intersects  $T_i$  precisely once. The class  $[c]$  does not lie in the image of map  $H_1(N; \mathbb{Z}) \rightarrow H_1(M; \mathbb{Z})$ , i.e. the map  $H_1(N; \mathbb{Z}) \rightarrow H_1(M; \mathbb{Z})$  is not surjective.

We now turn to the second case. So we suppose that there exists a  $W_i$  which is not contained in  $N$ . We say that a 3-manifold  $Y$  has *non-peripheral rational homology* if  $\text{coker}\{H_1(\partial Y; \mathbb{Q}) \rightarrow H_1(Y; \mathbb{Q})\}$  is non-trivial. It is straightforward to see that if  $Y$  has non-peripheral rational homology, then any finite cover of  $Y$  also has non-peripheral rational homology. By [AFW15, (C.17) and (H.13)] there exists a finite regular cover  $\widehat{W}_i$  of  $W_i$  that has non-peripheral rational homology.

By [WZ10, Theorem A] the profinite topology on the graph of groups defined by the  $T_i$ 's and  $W_j$ 's is efficient. (Strictly speaking [WZ10, Theorem A] is stated for the JSJ-decomposition of  $M$ , but an inspection of the proof shows that the statement holds for any decomposition of  $M$  along tori into Seifert fibered pieces and hyperbolic pieces.) This means in particular that the profinite topology on  $\pi_1(M)$  induces the profinite topology on  $\pi_1(W_i)$ . Therefore there exists a finite covering

$p: \widetilde{M} \rightarrow M$  such that each component of  $p^{-1}(W_i)$  covers  $\widehat{W_i}$ . In particular each component of  $p^{-1}(W_i)$  has non-peripheral rational homology. We denote by  $\widetilde{N}$  the corresponding cover of  $N$ . Note that  $\widetilde{N}$  does not contain  $p^{-1}(W_i)$ . Now it follows from a Mayer–Vietoris argument that  $\text{coker}\{H_1(\widetilde{N}; \mathbb{Q}) \rightarrow H_1(\widetilde{M}; \mathbb{Q})\}$  surjects onto  $\text{coker}\{H_1(\partial(p^{-1}(W_i)); \mathbb{Q}) \rightarrow H_1(p^{-1}(W_i); \mathbb{Q})\}$  which is non-zero. Thus, as desired, we found a finite cover such that  $H_1(\widetilde{N}; \mathbb{Z}) \rightarrow H_1(\widetilde{M}; \mathbb{Z})$  is not an isomorphism.

### 4.3. Proof of Corollary 1.6.

*Proof of Corollary 1.6.* Let  $f: M \rightarrow N$  be a proper map between two aspherical 3-manifolds with empty or toroidal boundary. We suppose that  $\widehat{\pi_1(M)} \cong \widehat{\pi_1(N)}$  and that  $f$  is a  $\pi_1$ -epimorphism.

*Claim.* The induced map  $f_*: \widehat{\pi_1(M)} \rightarrow \widehat{\pi_1(N)}$  is an isomorphism.

Given a finitely generated group  $\Gamma$  and  $n \in \mathbb{N}$  we denote by  $\Gamma(n)$  the intersection of the kernels of all homomorphisms from  $\Gamma$  to finite groups of order  $\leq n$ . Note that each  $\Gamma(n)$  is a finite-index characteristic subgroup. We write  $A = \pi_1(M)$  and  $B = \pi_1(N)$ . By the proof of [RZ10, Theorem 3.2.7] it suffices to prove that for every  $n$  the induced homomorphism  $f_*: A/A(n) \rightarrow B/B(n)$  is an isomorphism. Since  $f: A \rightarrow B$  is an epimorphism it follows that for every  $n$  the induced homomorphism  $A/A(n) \rightarrow B/B(n)$  is an epimorphism. On the other hand a straightforward consequence of Lemma 4.4 together with our assumption that  $\widehat{A} \cong \widehat{B}$  is that  $A/A(n)$  and  $B/B(n)$  are isomorphic. Since these groups are finite it now follows that for every  $n$  the map  $f_*: A/A(n) \rightarrow B/B(n)$  is an isomorphism. This concludes the proof of the claim.

Since  $f_*: \widehat{\pi_1(M)} \rightarrow \widehat{\pi_1(N)}$  is an isomorphism it follows that the map  $f_*: \pi_1(M) \rightarrow \pi_1(N)$  is a monomorphism. The Grothendieck rigidity of  $\pi_1(N)$ , proved in Theorem 1.5, implies that  $f: \pi_1(M) \rightarrow \pi_1(N)$  is an isomorphism. Since  $f$  is proper it follows as in the proof of Theorem 1.2 that  $f$  is homotopic to a homeomorphism.  $\square$

### REFERENCES

- [Ag08] I. Agol, *Criteria for virtual fibering*, J. Topol. 1 (2008), no. 2, 269–284.
- [Ag13] I. Agol, *The virtual Haken conjecture*, with an appendix by I. Agol, D. Groves and J. Manning, Documenta Math. 18 (2013), 1045–1087.
- [AF11] M. Aschenbrenner and S. Friedl, *Residual properties of graph manifold groups*, Top. Appl. 158 (2011), 1179–1191.
- [AFW15] M. Aschenbrenner, S. Friedl and H. Wilton, *3-manifold groups*, EMS Series of Lectures in Mathematics (2015)
- [BP92] R. Benedetti and C. Petronio, *Lectures on hyperbolic geometry*, Universitext, Springer-Verlag, Berlin (1992)
- [BCG95] G. Besson, G. Courtois and S. Gallot, *Entropies et rigidités des espaces localement symétriques de courbure strictement négative*, Geom. Funct. Anal. 5 (1995), no. 5, 731–799.
- [BW96] M. Boileau and S. Wang, *Non-zero degree maps and surface bundles over  $S^1$* , J. Differential Geom. 43 (1996), no. 4, 789–806.

- [CG87] A. Casson and C. McA. Gordon, *Reducing Heegaard splittings*, Topology and its Applications, 27 (1987), 275–283.
- [Ca12] W. Cavendish, *Finite-sheeted covering spaces and solenoids over 3-manifolds*, PhD thesis, Princeton University, 2012.
- [De13] J. DeBlois, *Explicit rank bounds for cyclic covers*, preprint (2013), arXiv:1310.7823. To appear in Algebraic and Geometric Topology.
- [DFV14] J. DeBlois, S. Friedl and S. Vidussi, *The rank gradient for infinite cyclic covers of 3-manifolds*, Michigan Math. J. 63 (2014), 65–81.
- [Der03] P. Derbez, *A criterion for homeomorphism between closed Haken manifolds*, Algebr. Geom. Topol. 3 (2003), 335–398.
- [Der10] P. Derbez, *Topological rigidity and Gromov simplicial volume*, Comment. Math. Helv. 85 (2010), no. 1, 1–37.
- [Der12] P. Derbez, *Local rigidity of aspherical three-manifolds*, Ann. Inst. Fourier (Grenoble) 62 (2012), no. 1, 393–416.
- [FV12] S. Friedl and S. Vidussi, *A Vanishing Theorem for Twisted Alexander Polynomials with Applications to Symplectic 4-manifolds*, J. Eur. Math. Soc. 16 (2013) no. 6, 2027–2041.
- [He76] J. Hempel, *3-Manifolds*, Ann. of Math. Studies, no. 86. Princeton University Press, Princeton, N. J., 1976.
- [He87] J. Hempel, *Residual finiteness for 3-manifolds*, Combinatorial group theory and topology (Alta, Utah, 1984), pp. 379–396, Ann. of Math. Stud., 111, Princeton Univ. Press, Princeton, NJ, 1987.
- [Gr82] M. Gromov, *Volume and bounded cohomology*, Publications mathématiques de I.H.S. 56 (1982), 5–99.
- [Ja80] W. Jaco, *Lectures on three-manifold topology*, CBMS Regional Conference Series in Mathematics, 43. American Mathematical Society, Providence, R.I. (1980)
- [JS79] W. Jaco and P. Shalen. *Seifert fibered spaces in 3-manifolds*, Mem. Amer. Math. Soc. 21 (1979), no. 220.
- [La04] M. Lackenby, *The asymptotic behavior of Heegaard genus*, Math. Res. Lett. 11 (2004), no. 2–3, 139–149.
- [La05] M. Lackenby, *Expanders, rank and graphs of groups*, Israel J. Math. 146 (2005), 357–370.
- [La06] M. Lackenby, *Heegaard splittings, the virtually Haken conjecture and Property  $(\tau)$* , Invent. math. 164 (2006), 317–359.
- [Liu013] Y. Liu, *Virtual cubulation of nonpositively curved graph manifolds*, J. Topol. 6 (2013), 793–822.
- [LR11] D. D. Long and A. Reid, *Grothendieck’s problem for 3-manifold groups*, Groups, Geometry and Dynamics, 5 (2011), 479–499.
- [Lü94] W. Lück, *Approximating  $L^2$ -invariants by their finite-dimensional analogues*, Geom. Funct. Anal. 4 (1994), 455–481.
- [Lü02] W. Lück,  *$L^2$ -Invariants: Theory and Applications to Geometry and K-Theory*, Ergebnisse der Mathematik und ihrer Grenzgebiete, 3. Folge, vol. 44, Springer, Berlin, 2002.
- [LW93] J. Luecke and Y.-Q. Wu, *Relative Euler number and finite covers of graph manifolds*, in: Geometric Topology, pp. 80103, AMS/IP Studies in Advanced Mathematics, vol. 2.1, Amer. Math. Soc., Providence, RI; International Press, Cambridge, MA, 1997.
- [Mo68] G. D. Mostow, *Quasi-conformal mappings in  $n$ -space and the rigidity of hyperbolic space forms*, Inst. Hautes Études Sci. Publ. Math. 34 (1968), 53–104.
- [Ne96] W. Neumann, *Commensurability and virtual fibration for graph manifolds*, Topology 39 (1996), 355–378.

- [NS07] N. Nikolov and D. Segal, *On finitely generated profinite groups I: Strong completeness and uniform bounds*, Ann. of Math. 165 (2007), 171–236.
- [PS99] B. Perron and P. Shalen, *Homeomorphic graph manifolds: A contribution to the  $\mu$  constant problem*, Topology Appl. 99 (1999), no. 1, 1–39.
- [Pr73] G. Prasad, *Strong rigidity of  $\mathbb{Q}$ -rank 1 lattices*, Invent. Math. 21 (1973), 255–286.
- [PW12] P. Przytycki and D. Wise, *Mixed 3-manifolds are virtually special*, Preprint (2012).
- [PW14] P. Przytycki and D. Wise, *Separability of embedded surfaces in 3-manifolds*, Compositio Mathematica 150 (2014), 1623–1630.
- [Rei15] A. Reid, *Profinite properties of discrete groups*, Proceedings of Groups St Andrews 2013, L.M.S. Lecture Note Series 242, Cambridge Univ. Press (2015), 73–104.
- [Ren10] C. Renard, *Gradients de Heegaard sous-logarithmiques d’une variété hyperbolique de dimension 3 et fibres virtuelles*, Actes du Séminaire Théorie Spectrale et Géométrie de Grenoble 29 (2010–2011), 97–131.
- [Ren14] C. Renard, *Detecting surface bundles in finite covers of hyperbolic closed 3-manifolds*, Trans. Amer. Math. Soc. 366 (2014), no. 2, 979–1027.
- [RZ10] L. Ribes and P. Zalesskii, *Profinite groups. Second edition*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics 40. Springer-Verlag, Berlin, 2010.
- [Sch02] M. Scharlemann, *Heegaard splittings of compact 3-manifolds*, Handbook of geometric topology, ed. by R. Daverman and R. Sherr, 921–953, North-Holland, Amsterdam, 2002.
- [Sco73] P. Scott, *Compact submanifolds of 3-manifolds*, J. London Math. Soc. (2) 7 (1973), 246–250.
- [Se97] J. P. Serre, *Galois Cohomology*, Springer, Berlin, 1997.
- [Som95] T. Soma, *A rigidity theorem for Haken manifolds*, Math. Proc. Cambridge Philos. Soc. 118 (1995), no. 1, p. 141–160.
- [Sou08] J. Souto, *The rank of the fundamental group of certain hyperbolic 3-manifolds fibering over the circle*, Boileau, Michel (ed.) et al., The Zieschang Gedenkschrift. Geometry and Topology Monographs 14 (2008), 505–518.
- [St62] J. Stallings, *On fibering certain 3-manifolds*, 1962 Topology of 3-manifolds and related topics (Proc. The Univ. of Georgia Institute, 1961) pp. 95–100 Prentice-Hall, Englewood Cliffs, N.J. (1962)
- [Th79] W.P. Thurston, *The geometry and topology of three-manifolds*, Princeton University Mathematics Department (1979).
- [Wa68] F. Waldhausen, *On irreducible 3-manifolds which are sufficiently large*, Ann. of Math. (2) 87 (1968), 56–88.
- [WZ10] H. Wilton and P. Zalesskii, *Profinite properties of graph manifolds*, Geom. Dedicata 147 (2010), 29–45.
- [Wi09] D. Wise, *The structure of groups with a quasi-convex hierarchy*, Electronic Res. Ann. Math. Sci 16 (2009), 44–55.
- [Wi12a] D. Wise, *The structure of groups with a quasi-convex hierarchy*, 189 pages, preprint (2012), downloaded on October 29, 2012 from <http://www.math.mcgill.ca/wise/papers.html>
- [Wi12b] D. Wise, *From riches to RAAGs: 3-manifolds, right-angled Artin groups, and cubical geometry*, CBMS Regional Conference Series in Mathematics,

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